Unstable Periodic Orbit Analysis of Histograms of Chaotic Time Series

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Abstract

Using the Lorenz equations, we have investigated whether unstable periodic orbits (UPOs) associated with a strange attractor may predict the occurrence of the robust sharp peaks in histograms of some experimental chaotic time series. Histograms with sharp peaks occur for the Lorenz parameter value $r = 60.0$ but not for $r = 28.0$, and the sharp peaks for $r = 60.0$ do not correspond to any single histogram derived from a UPO. Despite this lack of correspondence, we show that histograms derived from a chaotic time series can be accurately predicted by an escape-time weighting of UPO histograms.

05.45.+b, 05.70.Ln, 82.40.Bj 47.27.Cn,

Time series are often the only data available to quantify the complicated aperiodic dynamics in experimental chaotic systems such as convecting fluids, electronic circuits, and laser systems [1]. Time series derived from deterministic chaos exhibit structure under different statistics which serves to characterize the chaotic state, e.g. the time series of a laser experiment in a chaotic regime can have intensity histograms with sharp reproducible peaks [2]. Chaotic time series have also been used to detect unstable periodic orbits (UPOs) directly from experimental data [3]. Given a knowledge of these UPOs, one can characterize chaotic time series [4], control chaos [5], approximate natural measures [6], predict mean quantities [7, 8], and optimize performance functions in the system [9]. Unfortunately, it is a difficult task to detect sufficiently many UPOs directly from time series data to characterize chaos, causing the current reliance on the statistics of time series to characterize chaotic states.

The aim of this paper is to examine the relative success and limitations of trace formulas and escape-time weightings of UPOs in predicting the structure of histograms of chaotic time series data. To this end, the Lorenz equations [10] are well-suited as they can model a single-mode laser [11] and therefore be used to make qualitative comparisons with chaotic multimode laser experiments that have intensity histograms that exhibit sharp reproducible peaks [2]. The structure of the x-variable histogram of the Lorenz equations qualitatively changes from smooth for $r = 28.0$ to exhibiting sharp reproducible peaks at $r = 60.0$ and hence these histograms characterize the chaotic state. Further in using UPOs to characterize chaos, the Lorenz equations represents both ideal chaotic flows where the symbolic dynamics is understood ($r = 28.0$) and the more general non-ideal case where the symbolic dynamics of the flow is not understood ($r = 60.0$). We demonstrate that the structure of histograms of the chaotic x-variable can be predicted in terms of the UPOs of the Lorenz equations using trace formulas where a symbolic dynamics is understood and escape-time averages where a symbolic dynamics is not understood.

The importance of unstable periodic orbits in understanding Axiom-A chaotic systems has been well-known since the pioneering work of Smale [12]. Given a *complete* and *ordered* set of UPOs up to some symbolic length [13], trace formulas can be used to approximate the natural measure of Axiom-A chaotic attractors [6]. Although impressive work has been done in approximating averages using trace formulas and cycle expansions in low-dimensional chaotic maps [7, 14], much less is known about chaotic flows in ordinary and partial differential equations that may better describe many sorts of experimental chaotic systems. Statistical averages of UPOs based on trace formulas [6] and cycle expansions [14] developed in the context of low-dimensional chaotic maps become inapplicable to chaotic flows due to the lack of a well-understood symbolic dynamics. A unique symbolic dynamics is required both to determine a complete set of low-period UPOs and to order these UPOs. As an example, in the high-fractal-dimension $D = 8.8$ Kuramoto-Sivashinsky equation (where the symbolic dynamics is not understood) a simple escape-time weighting of UPOs proved superior to approximate trace formulas in predicting averages [8]. Even for low-dimensional chaotic flows, one must identify a complete symbolic dynamics of the flow and must compute a relatively complete set of UPOs or trace formulas will not converge. When these two conditions are not satisfied, the escape-time weighting of UPOs may prove superior in approximating averages of low-dimensional chaotic flows such as the structure in histograms of chaotic time series.

The escape-time weighting (ETW) approximates averages of chaos without a knowledge of symbolic dynamics and without complete ordered sets of UPOs [8]. ETW weights the average quantity m_j computed over UPO$_j$ by the escape-time τ_j,

$$\tau_j = \frac{1}{\sum_+ \lambda_{ij}} \tag{1}$$

where the above sum is over all positive Lyapunov exponents λ_{ij} of UPO$_j$. Since ETW is based on Lyapunov exponents, it is an **intensive** weighting which eliminates the necessity of ordering UPOs by symbolic length or period. Physically, the escape-time τ_j of a UPO is a time scale which reflects how long trajectories remain close to and track a UPO before the trajectory diverges due to the instability of the orbit. Escape-times of UPOs are typically shorter than a UPO period perhaps making ETW better suited for approximating averages in chaotic systems that do not exhibit near recurrences over typical experimental observation times. Given that sufficiently many UPOs are computed, the estimate of an average quantity m of chaos in terms of the same average quantities m_j over UPOs is given by,

$$m = \frac{\sum \tau_j m_j}{\sum \tau_j}. \tag{2}$$

Table I demonstrates that the convergence of ETW can be comparable to trace formulas [6] and zeta functions [7] even for low-dimensional maps with a known symbolic dynamics such as the logistic map. In addition, ETW is more effective in predicting the structure of histograms of chaotic time series as we discuss below.

To study the histogram structure of chaotic time series in terms of computed UPOs we consider the Lorenz equations [10],

$$\begin{aligned} dx/dt &= \sigma(y - x) \\ dy/dt &= rx - y - xz \\ dz/dt &= xy - bz \end{aligned} \tag{3}$$

where $\sigma = 10.0$, $b = 8/3$, and $r = 28.0$ or 60.0. The Lorenz equations were integrated with a 2nd-order accurate Adams-Bashforth method with a time step of $dt = 10^{-5}$ time units. To construct long-time histograms, the x-variable was recorded every time step over an integration time of 10^8 time units and the x-values were binned using 1000 equally spaced bins that spanned the range of x from -20.0 to 20.0 for $r = 28.0$ and from -30.0 to 30.0 for $r = 60.0$ (Fig. 1). Histograms were not sensitive to modest changes in the number of bins. The histograms of long time series of the x-variable can be either smooth ($r = 28.0$) or exhibit sharp reproducible peaks ($r = 60.0$). The number and position of the sharp peaks for Lorenz parameters $\sigma = 10.0$, $b = 8/3$, and $r = 60.0$ are robust in the presence of large amounts of multiplicative and additive noise suggesting that sharp reproducible peaks in histograms may exist for many sorts of experimental time series. For $r = 28.0$, short-time histograms (binning times $< 10^5$) exhibited random placements of peaks dependent on initial condition, while for $r = 60.0$ short-time histograms exhibited sharp peaks at the same characteristic x-values similar to the experimental findings of Bracikowski et al. [2].

To predict histograms such as those in Fig. 1 in terms of UPOs, a damped-Newton algorithm [8] was applied to the Lorenz equations to compute 664 UPOs for $r = 28.0$

and 599 UPOs for $r = 60.0$. The damped-Newton algorithm had a 95% success rate of computing a UPO given an initial guess which consisted of a point on the chaotic attractor and a randomly chosen period $T < 15.0$. Each newly computed UPO was compared with those already computed to form a unique set of non-repeated UPOs. The x-variable UPO histograms exhibit sharp peaks (Fig. 2) due to the turning points in the orbits. Empirically we observe that no single UPO histogram can predict the placement of peaks in the $r = 60.0$ long-time chaotic x-variable histogram. However, for binning times on the order of a UPO period, histograms of the chaotic x-variable often had similar placements of peaks as the UPO histograms (Fig. 2), providing evidence that the computed UPOs lie on the chaotic attractor.

As no single UPO could predict the smooth long-time histogram of the chaotic x-variable for $r = 28.0$, trace formula and escape-time averages using many UPOs were used to approximate the long-time histogram. In order to apply a trace formula, the set of computed UPOs at $r = 28.0$ were ordered using a complete binary symbolic dynamics determined by a Lorenz map [15]. Using symbolic dynamics, we determined that the computed set of $r = 28.0$ unstable periodic orbits was missing only seven orbits up to symbolic length $N = 10$. The $N = 10$ trace formula applied to x-variable UPO histograms (102 UPOs) resulted in an average histogram that well approximated the long-time x-variable histogram of the Lorenz attractor (Fig 3(b)). The escape-time weighting average of x-variable UPO histograms (using no ordering) was also satisfactory except at $x = 0$ (Fig 3(c)). The trace formula is superior to escape-time weighting for Lorenz parameter $r = 28.0$, because of the formula understanding of the symbolic dynamics of the flow. It is remarkable that sharp peaked x-variable UPO histograms can in both the trace formula and the escape-time weighting predict the smooth long-time histogram at Lorenz parameter $r = 28.0$. The great predictive power of averages of UPOs is demonstrated by this calculation in noting that the $N = 10$ trace formula used only 102 UPOs of period $T < 7.8$ time units to predict the smooth long-time histogram of the chaotic attractor which required 10^8 time units to construct!

At Lorenz parameter $r = 60.0$, the long-time x-variable histogram is qualitatively different than at $r = 28.0$ exhibiting sharp reproducible peaks at specific values of the x-variable. At $r = 60.0$ the UPOs are only approximately ordered by the binary symbolic dynamics of the Lorenz map [15, 16]. Without a unique symbolic dynamics it is impossible to determine whether we have computed a sufficiently complete and ordered set of UPOs on which to apply a trace formula. Using the approximate symbolic dynamics and hence an approximate ordering of UPOs, the trace formula using all UPO histograms of symbolic length $N = 10$ (118 UPOs) predicts an average histogram that does not accurately approximate the long-time histogram of the chaotic x-variable (Fig 4(b)). Trace formulas of other symbolic lengths also resulted in inaccurate approximations. Prior work using the same incomplete binary symbolic dynamics and trace formalisms to estimate the Hausdorff dimension at Lorenz parameter $r = 60.0$ demonstrated some success due to the insensitivity of this *particular average quantity* to missing cycles [16]. In general, approximate trace formulas will not succeed in accurately predicting averages as is demonstrated above by the $r = 60.0$ trace formula average histogram. Applying ETW to all *computed* x-variable UPO histograms resulted in an average histogram that predicts the peaks in the long-time histogram of the chaotic x-variable with incredible accuracy (Fig 4(c)) (the peak at $x = 0$ is due to a slight overweighting homoclinic orbits). ETW ignores the ordering of UPOs and is also less sensitive

to incomplete sets of UPOs (as is the case for the $r = 60.0$ data) making ETW preferable in averaging UPOs of chaotic flows.

Using trace formulas of unstable periodic orbits to predict averages in Axiom-A chaotic systems is well-established for low-dimensional chaotic flows which can be mapped onto a unique symbolic dynamics. The agreement between the trace formula average of UPO histograms at $r = 28.0$ and the long-time histogram of the chaotic x-variable is an example of the accuracy that can be achieved when a symbolic dynamics is understood. For general chaotic flows, trace formalisms will fail due to not understanding the symbolic dynamics of the UPOs. Not understanding the symbolic dynamics causes both an inability to order the UPOs and an inability to determine the completeness of a computed set of UPOs. Escape-time weighting is an intensive weighting **not** based on symbolic dynamics and so does not require any ordering of the UPOs making it preferable for approximating averages of chaotic flows when the geometry of the attractor is not understood. As an example, the escape-time average of x-variable UPO histograms at Lorenz parameter $r = 60.0$ predicts the reproducible peaks found in the long-time histogram of the chaotic x-variable which can not be predicted by trace formulas. Not having a unique symbolic dynamics and not having complete sets of UPOs will be general consequences of characterizing low- and high-dimensional chaotic flows making ETW a powerful method for extracting useful information from unstable periodic orbits of flows.

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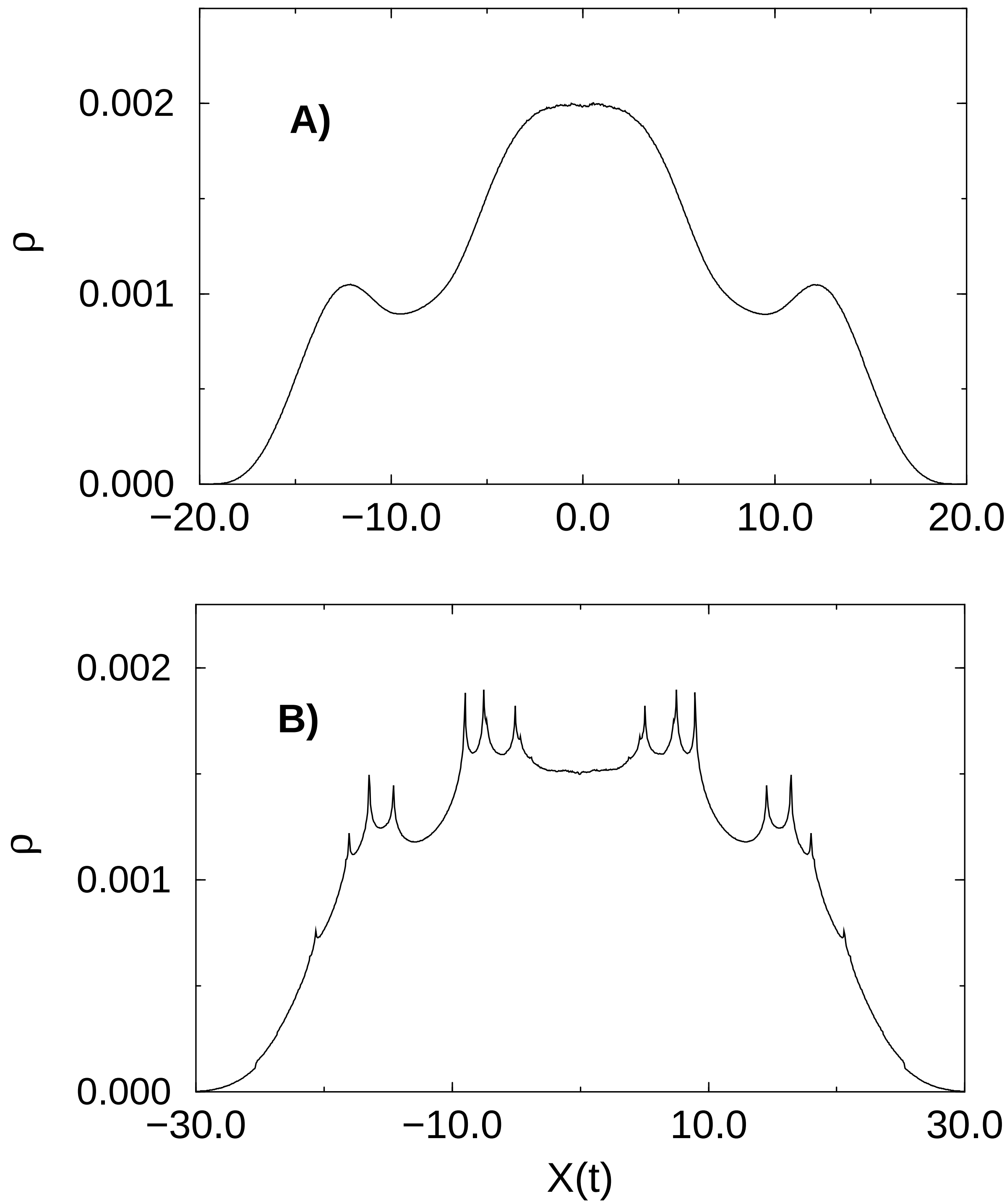
FIGURES

FIG. 1. Long-time x-variable histogram ρ of the Lorenz equations for **(A)**: Lorenz parameters $\sigma = 10.0,\ b = 8/3,$ and $r = 28.0$ and for **(B)**: Lorenz parameters $\sigma = 10.0,\ b = 8/3,$ and $r = 60.0$. Integration times of 10^8 time units were used to construct the histograms and 1000 bins were used to discretize the range of x between -20.0 to 20.0 in **(A)** and -30.0 to 30.0 in **(B)**.

FIG. 2. Three representative x-variable histograms ρ of UPOs of the Lorenz equations ($\sigma = 10.0,\ b = 8/3,$ and $r = 28.0$): **(A)**: histogram of UPO 011, **(B)**: histogram of UPO 1000, **(C)**: histogram of UPO 00010000000. 1000 bins were used to divide the range of x between -20.0 to 20.0

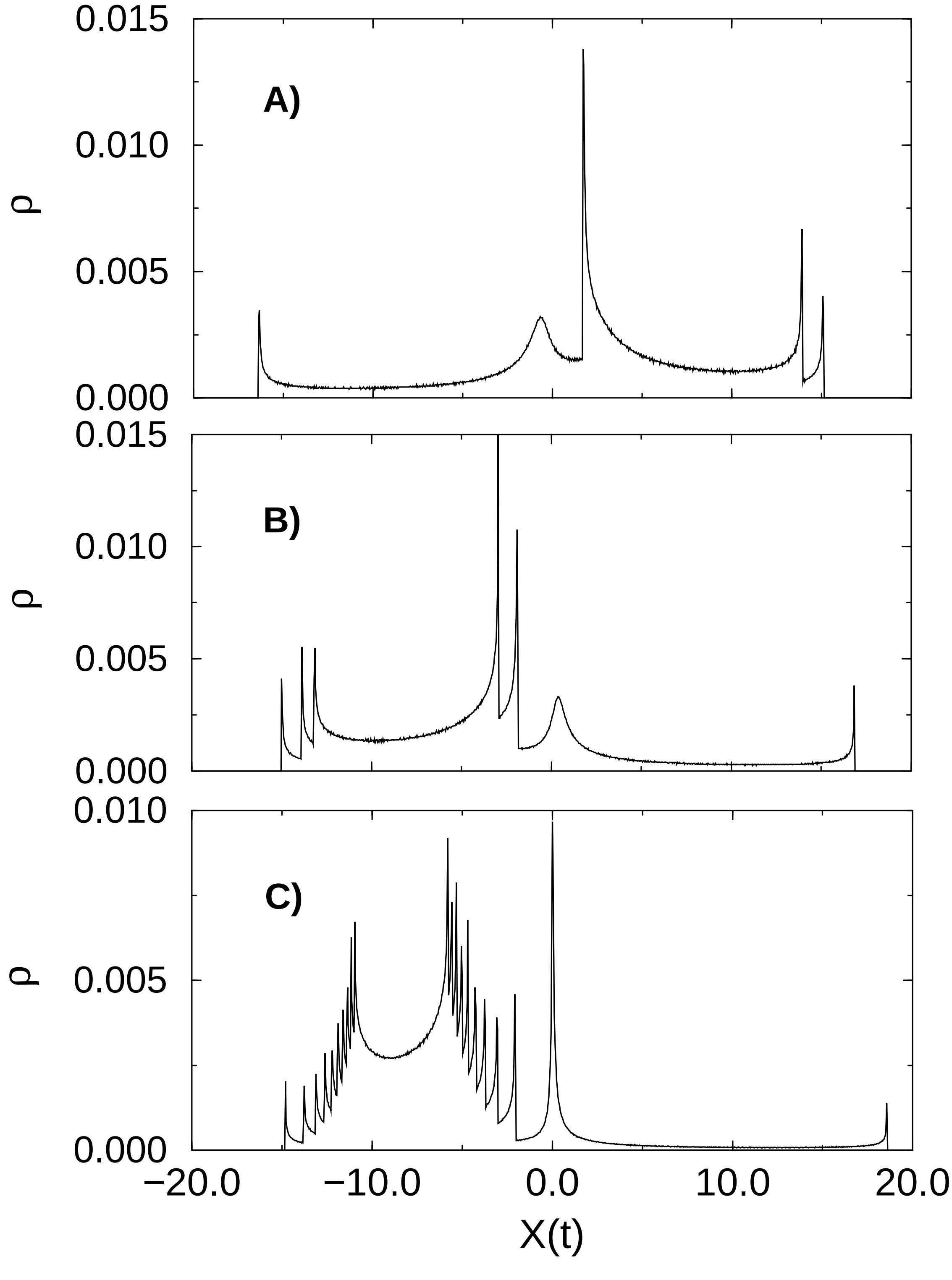
FIG. 3. **(A)**: Long-time x-variable histogram ρ of the Lorenz equations for Lorenz parameters $\sigma = 10.0,\ b = 8/3,$ and $r = 28.0$. **(B)**: Symbolic length $N = 10$ trace formula average histogram ρ_{TRACE} computed using 102 UPO histograms of the Lorenz equations. **(C)**: Escape-time weighting average histogram ρ_{ESCAPE} using 664 *computed* UPOs of the Lorenz equations. 1000 bins were used to divide the range of x between -20.0 to 20.0

FIG. 4. **(A)**: Long-time x-variable histogram ρ of the Lorenz equations for Lorenz parameters $\sigma = 10.0,\ b = 8/3,$ and $r = 60.0$. **(B)**: Symbolic length $N = 10$ trace formula average histogram ρ_{TRACE} computed using 118 UPO histograms of the Lorenz equations. The Lorenz map was used to construct a binary symbolic dynamics at $r = 60.0$, although not all UPOs had a unique symbolic designation. **(C)**: Escape-time weighting average histogram ρ_{ESCAPE} using 599 *computed* UPOs of the Lorenz equations. 1000 bins were used to divide the range of x between -30.0 to 30.0

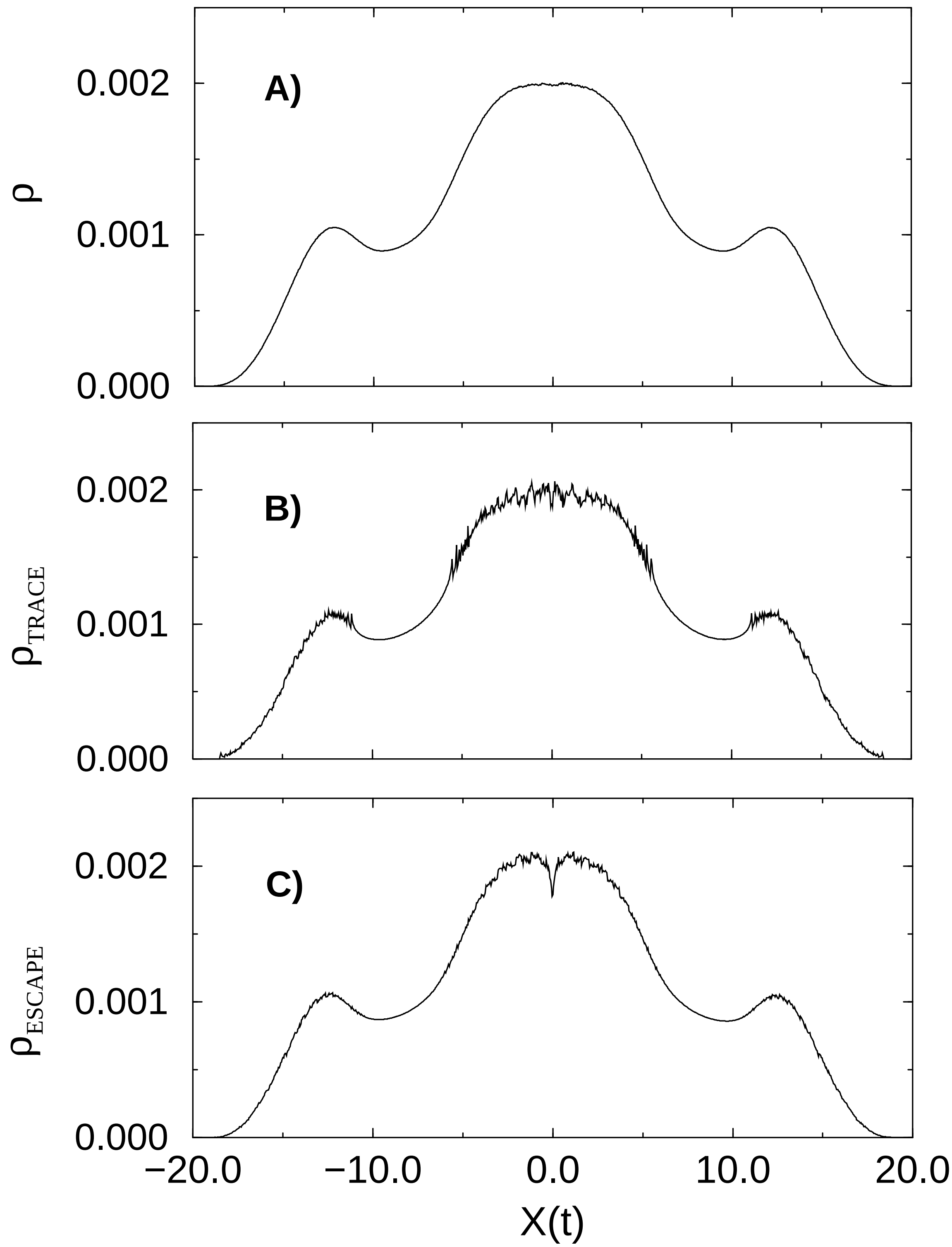


A)
ρ
0.002
0.001
0.000
−20.0
−10.0
0.0
10.0
20.0
B)
ρ
0.002
0.001
0.000
−30.0
−10.0
10.0
30.0
X(t)

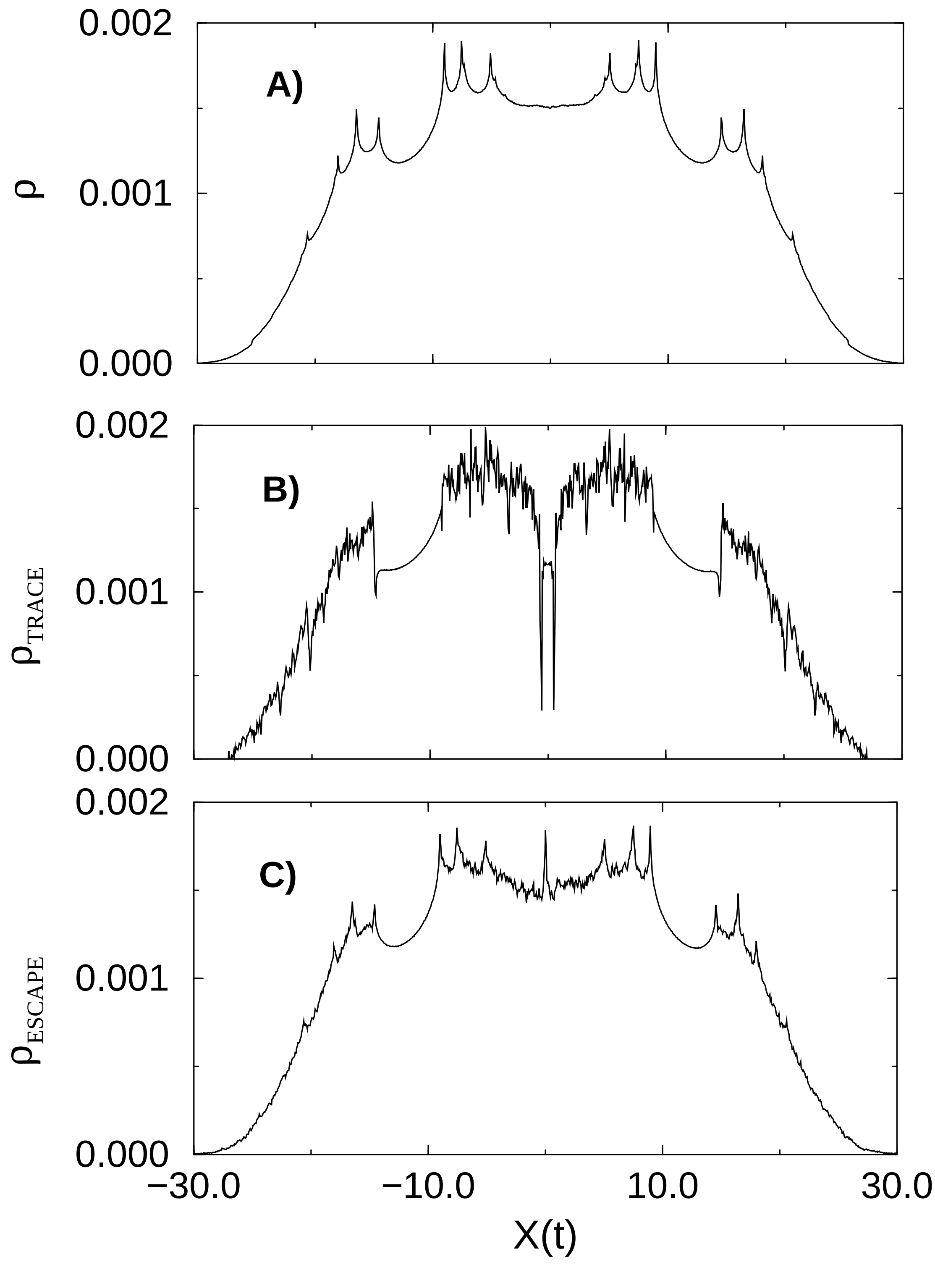
ZOLDI FIG1


0.015
0.010
0.005
0.000
A)
ρ
0.015
0.010
0.005
0.000
B)
ρ
0.010
0.005
0.000
C)
ρ
−20.0
−10.0
0.0
10.0
20.0
X(t)

ZOLDI FIG2



ZOLDI FIG3



ZOLDI FIG 4

TABLES

TABLE I. Approximations to the average of $< x^2 >= 0.375$ for the logistic map $x[i+1] = 4x[i](1 - x[i])$ using complete sets of low-period UPOs. N denotes the binary symbolic length of UPOs used in each approximation.

N	ETW	TRACE	ZETA
2	0.429	0.462	0.445
3	0.414	0.421	0.424
4	0.392	0.398	0.406